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FREE VIBRATIONS OF A CONCENTRIC CIRCULAR CONICAL BEAM WITH A CONCENTRIC CIRCULAR CYLINDRICAL BORE

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An analytical solution describing the free transverse vibration of a concentric circular conical tube with a concentric circular cylindrical bore has been derived for a variable domain which is characteristic for ballistic tubes. Eigenvalues and eigenvectors are discussed for a particular set of boundary conditions.

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1. INTRODUCTION

In the investigation of the vibrational response of a mortar tube system which is subjected to initial conditions, boundary constraints and applied actions with the normal mode method of dynamic analysis a nonlinear equation occurred for which no explicit solution could be found in the literature. Since this equation of motion describing the vibrational displacement of a conical ballistic tube with a concentric cylindric bore is inherent to many mortar as well as gun tube systems, an explicit analytical solution is being derived for a variable domain characteristic to ballistic tubes.

2. FORMULATION OF THE EQUATION FOR THE TRANSVERSE VIBRATIONS OF A CONICAL BALLISTIC TUBE WITH A CYLINDRIC BORE

The transverse vibrational displacement of a beam with arbitrary cross section under the influence of applied forces is governed by the equation of motion

$$[EI\, y''(x,t)]'' + \rho A\, \ddot{y}(x,t) = F(x,t), \tag{2.1}$$

where y is the lateral displacement, x the coordinate measured along the axis of the beam, t the time variable, E Young's modulus of elasticity, I the moment of inertia of the cross-sectional area with respect to x, ρ the density, A the cross-sectional area, and F(x,t) the acting force. The symbols ' and · denote the partial derivatives

$\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t}$, respectively.

Adopting the normal mode method of dynamic analysis the solution of the inhomogeneous equation for a given set of boundary conditions can be approximated to any degree of accuracy by superimposition of orthonormal functions representing eigen-solutions of the homogeneous equation to the specific boundary problems[1]. Hence the problem of deriving the solution to Eq. (2.1) for a set of boundary conditions is reduced to solving the homogeneous partial differential equation for the same set of boundary conditions.

$$[EI\, y''(x,t)]'' + \rho A\, \ddot{y}(x,t) = 0 \tag{2.2}$$

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[1] *S. Timoshenko, D. H. Young, W. Weaver, Vibration Problems in Engineering, 1974, G. Wiley and Sons, Inc., NY.*

Since flexural rigidity (EI) and mass unit length (ρA) are time independent for practically all gun tube vibrational problems, we can separate the variables by setting

$$y(x,t) = g(x) \sin(at+\alpha), \tag{2.3}$$

where a is the frequency of vibration and α is the phase angle, and obtain a homogeneous fourth-order ordinary differential equation for the displacement amplitude g(x):

$$[EIg'']'' - a^2\rho Ag = 0 \tag{2.4}$$

The fundamental system of solutions consists, in this case, of four linearly independent solutions, g_1, g_2, g_3, and g_4, which must be found for the construction of the general solution to Eq. (2.4):

$$g = \sum_{1}^{4} c_n g_n \tag{2.5}$$

The constants c_1, c_2, c_3, and c_4 must be determined to within an arbitrary constant in each particular case from the boundary conditions at the end of the beam. In general, the end boundary conditions for gun tube vibrational response problems are physical constraints on deflection y, slope y', bending moment [EIy''], and shearing force [EIy'']'. For example, at a fixed end the deflection and slope are equal to zero ($y = 0$, $y' = 0$); at a simply supported end the deflection and bending moment are equal to zero ($y' = 0$, $[EI\, y''] = 0$); and at a free end the bending moment and the shearing force both vanish ($[EIy''] = 0$, $[EIy'']' = 0$). At an interior boundary the continuity of deflection, slope, bending moment and shearing force must be preserved. Since these functions appear in the formulation of the eigenvalue problem, we will derive explicit analytic expressions for them concurrently.

In this study we are only interested in the case where Young's modulus of elasticity and the density are constant and the beam is a conical ballistic tube with a cylindric bore. The geometry of the beam is outlined in Figure 2.1.

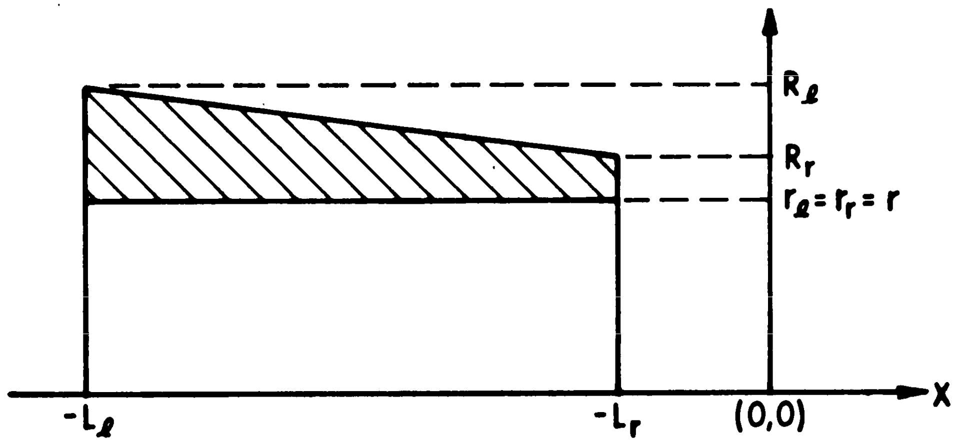

Figure 2.1. Geometry of the Concentric Circular Conic Beam with a Concentric Circular Cylindric Bore

From geometric considerations we can derive the following relations:

$$x = - L_r + \frac{R_\ell - R}{R_\ell - R_r} \left(- L_\ell + L_r \right) \text{ ....variable,} \tag{2.6}$$

$$EI = E \int_0^{2\pi} d\phi \int_r^R r^3 dr = \frac{E\pi}{2} \left(R^4 - r^4 \right) \text{ ....flexural rigidity,} \tag{2.7}$$

$$\rho A = \rho \int_0^{2\pi} d\phi \int_r^R r dr = \rho\pi \left(R^2 - r^2 \right) \text{ ....mass per unit length.} \tag{2.8}$$

Substituting Eqs. (2.6), (2.7), and (2.8) into Eq. (2.4), we obtain

$$\left[\left(\xi^4 - 1 \right) g'' \right]'' - \frac{2\rho}{Er^2} a^2 \left(\xi^2 - 1 \right) g = 0, \tag{2.9}$$

$$\xi = \frac{R_r}{r} + \left(\frac{R_\ell}{r} - \frac{R_r}{r} \right) \frac{x + L_r}{- L_\ell + L_r} .$$

3. DERIVATION OF THE FOUR LINEAR INDEPENDENT SOLUTIONS

The explicit form of Eq. (2.9) suggests the following coordinate transformation:

$$z = \xi - 1, \quad \frac{dz}{dx} = \sigma = \left(\frac{R_\ell}{r} - \frac{R_r}{r}\right) \frac{1}{-L_\ell + L_r} , \tag{3.1}$$

where the new variable z is now dimensionless and defined in the domain $\left[\frac{R_\ell}{r} - 1, \frac{R_r}{r} - 1\right]$. For most current weapon systems, the variable z, which is the ratio of wall thickness to bore radius of the ballistic tube, lies within the doman $0 \leq z < 1$. Therefore, the investigation will mainly be restricted to this geometric domain of interest. Carrying out the mapping of Eq. (2.9) from the x-space onto the z-space, we get

$$\left[\left(z^3 + 4z^2 + 6z + 4\right) z g_{zz}\right]_{zz} - \omega^4 (2 + z) z g = 0, \tag{3.2}$$

where the separation constant a is redefined as

$$a = k\omega^2 \quad \text{with } k = \sqrt{\frac{E}{2\rho}}\, r\sigma^2, \ \sigma = \left(\frac{R_\ell - R_r}{r}\right) \frac{1}{-L_\ell + L_r} . \tag{3.3}$$

The ordinary differential Eq. (3.2) has regular singular points at z = 0, -2, (-1 + i), (-1 - i), and an essential singularity at $z = \infty$. Since z = 0 is only a regular singularity, we can obtain a solution in the neighborhood of the origin by a series expansion. We try

$$g(z) = \sum_{\nu=0}^{\infty} a_\nu z^{n+\nu}, \ a_o \neq 0, \tag{3.4}$$

with the exponent n and all the coefficients a_ν still undetermined. By differentiating twice, we obtain

$$g_{zz} = \sum_{\nu=0}^{\infty} a_\nu (n + \nu)(n + \nu - 1) z^{n+\nu-2}. \tag{3.5}$$

By multiplying this expression by z^m and differentiating it again twice, we get

$$\left[z^m g_{zz}\right]_{zz} = \sum_{\nu=0}^{\infty} a_\nu \; (n + \nu)(n + \nu - 1)(n + m + \nu - 2) \tag{3.6}$$

$$\times (n + m + \nu - 3) z^{(n+m+\nu-4)}.$$

By substituting into Eq. (3.2), we have

$$\sum_{\nu=0}^{\infty} a_\nu \; (n + \nu)(n + \nu - 1) \; \Big[(n + \nu + 2)(n + \nu + 1) z^{n+\nu}$$

$$+4(n + \nu + 1)(n + \nu) z^{n+\nu-1} + 6(n + \nu)(n + \nu - 1) z^{n+\nu-2}$$

$$+4(n + \nu - 1)(n + \nu - 2) z^{n+\nu-3} \Big] - \omega^4 (2 + z) z \sum_{\nu=0}^{\infty} a_\nu z^{n+\nu} = 0. \tag{3.7}$$

The uniqueness theorem of power series requires that the coefficients of each power of z on the left hand of Eq. (3.7) must vanish individually. The lowest power of z appearing in Eq. (3.7) is z^{n-3} for $\nu = 0$. The requirement that the coefficient is zero yields

$$4n(n - 1)^2 (n - 2) a_o = 0. \tag{3.8}$$

Since a_o, by definition, is the coefficient of the lowest non-vanishing terms of the series, we have

$$n(n - 1)^2 (n - 2) = 0. \tag{3.8.1}$$

The roots of this indicial equation are n = 0, 1, 1, 2 resulting in three independent power series

$$g_1 = \sum_{\nu=0}^{\infty} a_{1\nu} z^\nu \quad , \quad g_2 = z \sum_{\nu=0}^{\infty} a_{2\nu} z^\nu \; , \; g_3 = z^2 \sum_{\nu=0}^{\infty} a_{3\nu} z^\nu. \tag{3.9}$$

Since n = 1 is a double root, a fourth linearly independent solution may be obtained by multiplying the regular solution $g_2(z)$ by a logarithmic term and adding another power series:

$$g_4(z) = g_2(z)\ln z + \sum_{\nu=0}^{\infty} a_{4\nu} z^{\nu+n}. \tag{3.10}$$

For the eigenvalue problem we need, in addition to the deflection g(z), the slope g_z, the bending moment $[EIg_{zz}]$, and the shearing force $[EIg_{zz}]_z$. To directly obtain explicit expressions for these physical functions, we will slightly modify the approach. Integration of Eq. (3.2) leads to

$$\left[\left(z^3 + 4z^2 + 6z + 4\right)zg_{zz}\right]_z = K_1 + \omega^2 \int dzz(2 + z)g(z) \tag{3.11}$$

$$= K_1^n + \omega^4 \sum_0^{\infty} a_{n\nu} \left[\frac{2}{k_n + 2 + \nu} + \frac{z}{k_n + 3 + \nu}\right] z^{k_n+2+\nu} \tag{3.11.1}$$

...for n = 1, 2, 3 and k_n = 0, 1, 2.

$$= K_1^4 + \omega^4 \left\{ \ln z \sum_0^{\infty} a_{2\nu} \left[\frac{2}{k_2 + 2 + \nu} + \frac{z}{k_2 + 3 + \nu}\right] z^{k_2+2+\nu} \right.$$

$$- \sum_0^{\infty} a_{2\nu} \left[\frac{2}{\left(k_2 + 2 + \nu\right)^2} + \frac{z}{\left(k_2 + 3 + \nu\right)^2}\right] z^{k_2+2+\nu}$$

$$\left. + \sum_0^{\infty} a_{4\nu} \left[\frac{2}{k_4 + 2 + \nu} + \frac{z}{k_4 + 3 + \nu}\right] z^{k_4+2+\nu} \right\} \tag{3.11.2}$$

for n = 4 and k_4 to be determined.

Integrating the above equation again, we have

$$\left[\left(z^3 + 4z^2 + 6z + 4\right)zg_{zz}\right] = K_o + K_1 z + \omega^4 \int dz \int dz'z' (2 + z') g(z'). \tag{3.12}$$

Substituting the power series Eqs. (3.9) and (3.10) into this equation, we have

$$\sum_0^\infty a_{n\nu}\left(k_n + \nu\right)\left(k_n + \nu - 1\right)\left(z^3 + 4z^2 + 6z + 4\right)z^{k_n+\nu-1}$$

$$= K_o^n + K_1^n z + \omega^4 \sum_0^\infty a_{n\nu} \frac{1}{k_n + 3 + \nu}\left[\frac{2}{k_n + 2 + \nu}\right.$$

$$\left. + \frac{z}{k_n + 4 + \nu}\right] z^{k_n+\nu+3} \qquad (3.12.1)$$

for $n = 1, 2, 3$ and $k_n = 0, 1, 2$;

$$\ln z \sum_0^\infty a_{2\nu}\left(k_2 + \nu\right)\left(k_2 + \nu - 1\right)\left(z^3 + 4z^2 + 6z + 4\right)z^{k_2+\nu-1}$$

$$+ \sum_0^\infty \left\{a_{2\nu}\left[2\left(k_2 + \nu\right) - 1\right]z^{k_2-k_4} + a_{4\nu}\left(k_4 + \nu\right)\left(k_4 + \nu - 1\right)\right\}$$

$$\times \left(z^3 + 4z^2 + 6z + 4\right)z^{k_4+\nu-1}$$

$$= K_o^4 + K_1^4 z + \omega^4 \left\{\ln z \sum_0^\infty a_{2\nu} \frac{1}{k_2 + 3 + \nu}\left[\frac{2}{k_2 + 2 + \nu}\right.\right.$$

$$\left. + \frac{z}{k_2 + 4 + \nu}\right] z^{k_2+\nu-3} + \sum_0^\infty \left\{-a_{2\nu} \frac{1}{k_2 + 3 + \nu}\left[\frac{2}{k_2 + 2 + \nu}\right.\right.$$

$$\times \left(\frac{1}{k_2 + 2 + \nu} + \frac{1}{k_2 + 3 + \nu}\right) + \frac{z}{k_2 + 4 + \nu}\left(\frac{1}{k_2 + 3 + \nu}\right.$$

$$\left.\left. + \frac{1}{k_2 + 4 + \nu}\right)\right] z^{k_2-k_4} + a_{4\nu} \frac{1}{k_4 + 3 + \nu}\left[\frac{2}{k_4 + 2 + \nu}\right.$$

$$\left.\left.\left. + \frac{z}{k_4 + 4 + \nu}\right]\right\} z^{k_4+\nu+3}\right\} \qquad (3.12.2)$$

for n = 4 and k_4 to be determined.

The uniqueness of power series requires that the total coefficient of each power of z vanishes all by itself. For the indicial equation roots n = 1, 2, and 3, we have

z^0: $\quad -K_0^1 = 0; \quad -K_0^2 = 0; \quad -K_0^3 = 0$

$$\longrightarrow K_0^n = 0;\ n = 1,\ 2,\ 3$$

z^1: $\quad -K_1^1 + 2.1.4a_{12} = 0; \quad -K_1^2 + 2.1.4a_{21} = 0;$

$$-K_1^3 + 2.1.4a_{30} = 0$$

$$\longrightarrow K_1^n = 8a_{n,3-n};\quad n = 1,2,3$$

z^2: $\quad 2.1.6a_{12} + 3.2.4a_{13} = 0$

$$2.1.6a_{21} + 3.2.4a_{22} = 0$$

$$2.1.6a_{30} + 3.2.4a_{31} = 0$$

$$\longrightarrow a_{n,4-n} = -\frac{1}{2}\,a_{n,3-n};\ n=1,2,3$$

z^3: $\quad 2.1.4a_{12} + 3.2.6a_{13} + 4.3.4a_{14} - \frac{\omega^4}{3.2} \cdot 2a_{10} = 0$

$$2.1.4a_{21} + 3.2.6a_{22} + 4.3.4a_{23} = 0$$

$$2.1.4a_{30} + 3.2.6a_{31} + 4.3.4a_{32} = 0$$

(Equations continued on next page)

$$\longrightarrow a_{14} = -\frac{1}{4.3}\left[\frac{3}{2}\cdot 3.2a_{13} + 2.1a_{12} - \frac{1}{3.2}\left(\frac{\omega^4}{4}\right)\cdot 2a_{10}\right]$$

$$a_{n,5-n} = -\frac{1}{4.3}\left[\frac{3}{2}\cdot 3.2a_{n,4-n} + 2.1a_{n,3-n}\right] \; ; \; n=2,3$$

$$z^4: \quad 2.1a_{12} + 3.2.4a_{13} + 4.3.6a_{14} + 5.4.4a_{15} - \frac{\omega^4}{4.3}\left(2a_{11} + a_{10}\right) = 0$$

$$2.1a_{21} + 3.2.4a_{22} + 4.3.6a_{23} + 5.4.4a_{24} - \frac{\omega^4}{4.3}\cdot 2a_{20} = 0$$

$$2.1a_{30} + 3.2.4a_{31} + 4.3.6a_{32} + 5.4.4a_{33} = 0$$

$$\longrightarrow a_{15} = -\frac{1}{5.4}\left[\frac{3}{2}\cdot 4.3a_{14} + 3.2a_{13} + \frac{1}{4}\cdot 2.1a_{12} - \frac{1}{4.3}\right.$$

$$\left.\times\left(\frac{\omega^4}{4}\right)\left(2a_{11} + a_{10}\right)\right]$$

$$a_{24} = -\frac{1}{5.4}\left[\frac{3}{2}\cdot 4.3a_{23} + 3.2a_{22} + \frac{1}{4}\cdot 2.1a_{21} - \frac{1}{4.3}\right.$$

$$\left.\times\left(\frac{\omega^4}{4}\right) 2a_{20}\right]$$

$$a_{33} = -\frac{1}{5.4}\left[\frac{3}{2}\cdot 4.3a_{32} + 3.2a_{31} + \frac{1}{4}\cdot 2.1a_{30}\right]$$

$$z^5: \quad 3.2a_{13} + 4.3.4a_{14} + 5.4.6a_{15} + 6.5.4a_{16} - \frac{\omega^4}{5.4}\left(2a_{12} + a_{11}\right) = 0$$

$$3.2a_{22} + 4.3.4a_{23} + 5.4.6a_{24} + 6.5.4a_{25} - \frac{\omega^4}{5.4}\left(2a_{21} + a_{20}\right) = 0$$

$$3.2a_{31} + 4.3.4a_{32} + 5.4.6a_{33} + 6.5.4a_{34} - \frac{\omega^4}{5.4}\cdot 2a_{30} = 0$$

$$\longrightarrow a_{n,7-n} = -\frac{1}{6.5}\left[\frac{3}{2}\cdot 5.4a_{n,6-n} + 4.3a_{n,5-n} + \frac{1}{4}\cdot 3.2a_{n,4-n}\right.$$

(Equations continued on next page)

$$- \frac{1}{5.4}\left(\frac{\omega^4}{4}\right)\left(2a_{n,3-n} + a_{n,2-n}\right)\Bigg] \; ; \quad n = 1, 2$$

$$a_{34} = -\frac{1}{6.5}\left[\frac{3}{2} \cdot 5.4a_{33} + 4.3a_{32} + \frac{1}{4} \cdot 3.2a_{31} - \frac{1}{5.4}\right.$$

$$\left. \times \left(\frac{\omega^4}{4}\right) 2a_{30}\right]$$

$$z^6: \quad 4.3a_{14} + 5.4.4a_{15} + 6.5.6a_{16} + 7.6.4a_{17} - \frac{\omega^4}{6.5}\left(2a_{13} + a_{12}\right) = 0$$

$$4.3a_{23} + 5.4.4a_{24} + 6.5.6a_{25} + 7.6.4a_{26} - \frac{\omega^4}{6.5}\left(2a_{22} + a_{21}\right) = 0$$

$$4.3a_{32} + 5.4.4a_{33} + 6.5.6a_{34} + 7.6.4a_{35} - \frac{\omega^4}{6.5}\left(2a_{31} + a_{30}\right) = 0$$

$$\longrightarrow \quad a_{n,8-n} = -\frac{1}{7.6}\left[\frac{3}{2} \cdot 6.5a_{n,7-n} + 5.4a_{n,6-n}\right.$$

$$\left. + \frac{1}{4} \cdot 4.3a_{n,5-n} - \frac{1}{6.5}\left(\frac{\omega^4}{4}\right)\left(2a_{n,4-n} + a_{n,3-n}\right)\right] \; ;$$

$$n = 1, 2, 3$$

$$\ldots \qquad (3.13)$$

Without loss of generality we can set the coefficients a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, and a_{23} to zero. By inspection and mathematical induction we obtain the following recurrence relations for the coefficients:

$$K_0^1 = K_0^2 = K_0^3 = 0$$

$$K_1^1 = K_1^2 = 0, \quad K_1^3 = 8a_{30}$$

$$a_{10} \neq 0, \quad a_{11} = a_{12} = a_{13} = 0, \quad a_{14} = \frac{1}{36}\left(\frac{\omega^4}{4}\right)$$

(Equations continued on next page)

$$a_{20} \neq 0, \quad a_{21} = a_{22} = a_{23} = 0, \quad a_{24} = \frac{1}{120}\left(\frac{\omega^4}{4}\right)$$

$$a_{30} \neq 0, \quad a_{31} = -\frac{1}{2}, \quad a_{32} = \frac{5}{24}, \; a_{33} = -\frac{1}{16}; \quad a_{34} = \frac{1}{240} + \frac{1}{300}\left(\frac{\omega^4}{4}\right)$$

$$a_{n,\nu+1} = -\frac{1}{(n+\nu)\,(n+\nu-1)}\left[\frac{3}{2}(n+\nu-1)\,(n+\nu-2)a_{n,\nu}\right.$$

$$+ (n+\nu-2)(n+\nu-3)a_{n,\nu-1} + \frac{1}{4}(n+\nu-3)\,(n+\nu-4)$$

$$\left.\times\, a_{n,\nu-2} - \frac{1}{(n+\nu-1)\,(n+\nu-2)}\left(\frac{\omega^4}{4}\right)\left(2a_{n,\nu-3} + a_{n,\nu-4}\right)\right]$$

$$\ldots \text{ for } n = 1, 2, 3 \text{ and } \nu = 4, 5, 6, \ldots. \qquad (3.14)$$

To derive the irregular solution, we apply the uniqueness theorem of power series to Eq. (3.12.2). Because the logarithmic terms vanish, we have

$$z^0: \quad -4.1a_{20} + K_0 = 0$$

$$\longrightarrow K_0 = 4a_{20}$$

$$z^1: \quad -6.1a_{20} - 4.3a_{21} + K_1 = 0$$

$$\longrightarrow K_1 = 6a_{20}$$

$$z^2: \quad -4.1a_{20} - 6.3a_{21} - 4.5a_{22}$$

This expression is $\neq 0$; to cancel it we must require that

$$k_4 = 3.$$

$$-4.1a_{20} - 6.3a_{21} - 4.5a_{22} - 4.3.2a_{40} = 0$$

(Equations continued on next page)

$$\longrightarrow \quad a_{40} = -\frac{1}{6}a_{20}$$

z^3:
$$-1.1a_{20} - 4.3a_{21} - 6.5a_{22} - 4.7a_{23} - 6.3.2a_{40} - 4.4.3a_{41} = 0$$

$$\longrightarrow \quad a_{41} = \frac{5}{48}a_{20}$$

z^4:
$$-3a_{21} - 4.5a_{22} - 6.7a_{23} - 4.9a_{24} - \frac{7\omega^4}{(3.4)^2}2a_{20} - 4.3.2a_{40}$$

$$-6.4.3a_{41} - 4.5.4a_{42} = 0$$

$$\longrightarrow \quad a_{42} = -\left[\frac{7}{160} + \frac{31}{3600}\left(\frac{\omega^4}{4}\right)\right]a_{20}$$

z^5:
$$-5a_{22} - 4.7a_{23} - 6.9a_{24} - 4.11a_{25} - \frac{9\omega^4}{(4.5)^2}\left(2a_{21} + a_{20}\right)$$

$$-3.2a_{40} - 4.4.3a_{41} - 6.5.4a_{42} - 4.6.5a_{43} = 0$$

$$\longrightarrow \quad a_{43} = -\frac{1}{6.5}\cdot\left\{\left[\frac{3}{2}\cdot 5.4a_{42} + 4.3a_{41} + \frac{1}{4}\cdot 3.2a_{40}\right]\right.$$
$$+\left[11a_{25} + \frac{3}{2}\cdot 9a_{24} + 7a_{23} + \frac{1}{4}\cdot 5a_{22} + \frac{9}{(4.5)^2}\right.$$
$$\left.\left.\times\left(\frac{\omega^4}{4}\right)\left(2a_{21} + a_{20}\right)\right]\right\}$$

z^6:
$$-7a_{23} - 4.9a_{24} - 6.11a_{25} - 4.13a_{26} - \frac{11\omega^4}{(5.6)^2}\left(2a_{22} + a_{21}\right)$$

$$-4.3a_{41} - 4.5.4a_{42} - 6.6.5a_{43} - 4.7.6a_{44} + \frac{\omega^4}{6.5}\cdot 2a_{40} = 0$$

(Equations continued on next page)

$$
\begin{aligned}
a_{44} = -\frac{1}{7.6}\Bigg\{\Bigg[&\frac{3}{2}\cdot 6.5a_{43} + 5.4a_{42} + \frac{1}{4}\cdot 4.3a_{41} \\
&- \frac{1}{6.5}\left(\frac{\omega^4}{4}\right) 2a_{40}\Bigg] + \Bigg[13a_{26} + \frac{3}{2}\cdot 11a_{25} + 9a_{24} \\
&+ \frac{1}{4}\cdot 7a_{23} + \frac{11}{(5.6)^2}\left(\frac{\omega^4}{4}\right)\left(2a_{22} + a_{21}\right)\Bigg]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
z^7: \quad & -9a_{24} - 4.11a_{25} - 6.13a_{26} - 4.15a_{27} - \frac{13\ \omega^4}{(6.7)^2}\left(2a_{23} + a_{22}\right) \\
& -5.4a_{42} - 4.6.5a_{43} - 6.7.6a_{44} - 4.8.7a_{45} + \frac{\omega^4}{7.6}\left(2a_{41} + a_{40}\right) = 0
\end{aligned}
$$

$$
\begin{aligned}
a_{45} = -\frac{1}{8.7}\Bigg\{\Bigg[&\frac{3}{2}\cdot 7.6a_{44} + 6.5a_{43} + \frac{1}{4}\cdot 5.4a_{42} \\
&- \frac{1}{7.6}\left(\frac{\omega^4}{4}\right)\left(2a_{41} + a_{40}\right)\Bigg] + \Bigg[15a_{27} + \frac{3}{2}\cdot 13a_{26} \\
&+ 11a_{25} + \frac{1}{4}\cdot 9a_{24} + \frac{13}{(6.7)^2}\left(\frac{\omega^4}{4}\right)\left(2a_{23} + a_{22}\right)\Bigg]\Bigg\}
\end{aligned}
$$

$$\ldots \tag{3.15}$$

From this pattern we can deduce the general recurrence formula:

$$
\begin{aligned}
a_{4,\nu+1} = -\frac{1}{(4+\nu)(3+\nu)}\Bigg\{\Bigg[&\frac{3}{2}(3+\nu)(2+\nu)a_{4,\nu} + (2+\nu)(1+\nu) \\
&\times a_{4,\nu-1} + \frac{1}{4}(1+\nu)\nu\, a_{4,\nu-2} - \frac{1}{(3+\nu)(2+\nu)}\left(\frac{\omega^4}{4}\right) \\
&\times\left(2a_{4,\nu-3} + a_{4,\nu-4}\right)\Bigg] + \Bigg[(2\nu+7)\, a_{2,\nu+3} + \frac{3}{2}(2\nu+5)a_{2,\nu+2} \\
&+ (2\nu+3)\, a_{2,\nu+1} + \frac{1}{4}(2\nu+1)a_{2,\nu} + \frac{(2\nu+5)}{(\nu+3)^2(\nu+2)^2}\left(\frac{\omega^4}{4}\right) \\
&\times\left(2a_{2,\nu-1} + a_{2,\nu-2}\right)\Bigg]\Bigg\}
\end{aligned}
\tag{3.16}
$$

To ascertain that Frobenius' method yields series solutions which not only satisfy our fourth-order differential equation but also converge over the region of interest ($0 \leq z \leq 1$), we have to determine the convergence of the series. The recurrence formulae (3.14) and (3.16) exhibit a $\left(\frac{\omega^4}{4}\right)$ dependence for the coefficients $a_{n,\nu}$ which is of the following structure:

$$a_{n0},\ a_{n1},\ a_{n2},\ a_{n3} \sim \left(\frac{\omega^4}{4}\right)^0$$

$$a_{n4},\ a_{n5},\ a_{n6},\ a_{n7} \sim \left(\frac{\omega^4}{4}\right)^1$$

$$a_{n8},\ a_{n9},\ a_{n10},\ a_{n11} \sim \left(\frac{\omega^4}{4}\right)^2$$

$$\cdots \tag{3.17}$$

Therefore, we will apply the D'Alembert ratio test to the ratio of the coefficients $a_{n,\nu+4}$ and $a_{n,\nu}$. For large ν, we can approximate the recurrence formulae for the regular solutions, Eq. (3.14), and for the nonlogarithmic part of the irregular solution, Eq. (3.5), by

$$a_{n,\nu+4} = -\frac{3}{2}\, a_{n,\nu+3} - a_{n,\nu+2} - \frac{1}{4}\, a_{n,\nu+1} + 0\left(\frac{1}{\nu}\right) \tag{3.18}$$

In order to deduce a recurrence formula which contains only terms proportional to $a_{n,\nu+4}$, $a_{n,\nu}$, $a_{n,\nu-4}$, ...we will use the recurrence formula for the (ν+3)-th term in addition to that of (ν+4)-th term.

$$a_{n,\nu+3} = -\frac{3}{2}\, a_{n,\nu+2} - a_{n,\nu+1} - \frac{1}{4}\, a_{n,\nu} + 0\left(\frac{1}{\nu}\right) \tag{3.19}$$

Addition and subtraction of Eqs. (3.17) and (3.18) yield

$$a_{n,\nu+3} = -\frac{4}{5}\, a_{n,\nu+4} - \frac{1}{2}\, a_{n,\nu+2} + \frac{1}{20}\, a_{n,\nu} + 0\left(\frac{1}{\nu}\right) \tag{3.20}$$

$$a_{n,\nu+1} = \frac{4}{5}\, a_{n,\nu+4} - a_{n,\nu+2} - \frac{3}{10}\, a_{n,\nu} + 0\left(\frac{1}{\nu}\right) . \tag{3.21}$$

These relations still contain the coefficient $a_{n,\nu+2}$. To eliminate the $a_{n,\nu+2}$ dependency, we apply relations (3.20) and (3.21) to the $[(\nu+3)-2,(\nu+1)-2]$ and $[(\nu+3)-4,(\nu+1)-4]$-th coefficients.

$$a_{n,\nu+1} = -\frac{4}{5}a_{n,\nu+2} - \frac{1}{2}a_{n,\nu} + \frac{1}{20}a_{n,\nu-2} + 0\left(\frac{1}{\nu}\right) \tag{3.22}$$

$$a_{n,\nu-1} = \frac{4}{5}a_{n,\nu+2} - a_{n,\nu} - \frac{3}{10}a_{n,\nu-2} + 0\left(\frac{1}{\nu}\right) \tag{3.23}$$

$$a_{n,\nu-1} = -\frac{4}{5}a_{n,\nu} - \frac{1}{2}a_{n,\nu-2} + \frac{1}{20}a_{n,\nu-4} + 0\left(\frac{1}{\nu}\right) \tag{3.24}$$

$$a_{n,\nu-3} = \frac{4}{5}a_{n,\nu} - a_{n,\nu-2} - \frac{3}{10}a_{n,\nu-4} + 0\left(\frac{1}{\nu}\right) \tag{3.25}$$

By successive substitution of Eqs. (3.21), (3.22), (3.23), and (3.24) and simple manipulation we obtain the following identity in the form of a limit:

$$\lim_{\nu\to\infty}\left\{\frac{a_{n,\nu+4}}{a_{n,\nu}}\cdot\frac{a_{n,\nu}}{a_{n,\nu-4}} + \frac{3}{16}\frac{a_{n,\nu}}{a_{n,\nu-4}} - \frac{1}{64} + 0\left(\frac{1}{\nu}\right)\right\} = 0 \tag{3.26}$$

Setting

$$R = \lim_{\nu\to\infty}\left(\frac{a_{n,\nu+4}}{a_{n,\nu}}\right) = \lim_{\nu\to\infty}\left(\frac{a_{n,\nu}}{a_{n,\nu-4}}\right) \tag{3.27}$$

and taking the limit as $\nu\to\infty$, we derive a quadratic equation for the ratio of the coefficients:

$$R^2 + \frac{3}{16}R - \frac{1}{64} = 0 \ . \tag{3.28}$$

Its roots are

$$R = -\frac{1}{4} \text{ and } \frac{1}{16} \ . \tag{3.29}$$

Using these values, we can establish a lower bound of the radius of convergence for the series solution as

$$r^* = \min \left(|R_1|^{-\frac{1}{4}}, \quad |R_2|^{-\frac{1}{4}} \right) = \min (\sqrt{2}, 2) = \sqrt{2} \quad , \tag{3.30}$$

which is equal to the distance between the point $z = 0$ and the regular singular points $z = (-1 \pm i)$. Since our power series, Eqs (3.9) and (3.10) are of the form

$$g_n(z) = z^{n-1} \sum_{0}^{\infty} a_{n,\nu} z^{\nu} + \delta_{4n} (z \ln z) \sum_{0}^{\infty} a_{2,\nu} z^{\nu} \tag{3.31}$$

$$(n = 1, 2, 3, 4; \ \delta_{4n} \ldots \text{Kronecker delta function});$$

they converge at the point $z = 0$. The power series solution to Eq. (3.2),

$$g(z) = \sum_{1}^{4} \gamma_n g_n(z), \quad (\gamma_n \ldots \text{ arbitrary constants}), \tag{3.32}$$

is convergent for $z = re^{i\theta}$, $0 \leq r < r^*$, $-\pi < \theta \leq \pi$, in the z-plane cut along the negative real axis and uniformly and absolutely convergent for any interior domain, $r \leq r^{**}$, where $0 \leq r^{**} < r^* = \sqrt{2}$.

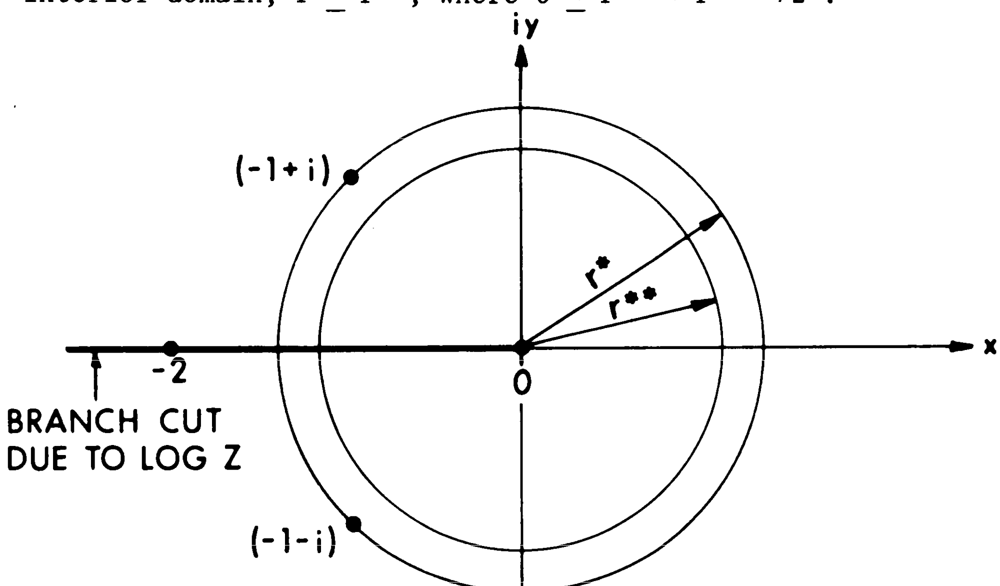

Figure 3.1. Domain of Convergence for g(z).

$$\left[z = x + iy = re^{i\theta}, \ r = \sqrt{x^2 + y^2} \ , \ \theta = \text{arctg} \left(\frac{y}{x} \right) \right]$$

Since each of the terms $g_{n\nu}(z) = a_{n\nu} z^{n+\nu} + \delta_{4n}(z \ln z)\, a_{2\nu} z^n$, $n = 1, 2, 3, 4$, is a continuous function of z and the partial solutions g_n, $n = 1, 2, 3, 4$, converge uniformly for $|z| \leq r^{**}$, the power series solution $g(z)$, Eq. (3.31), is a continuous function in our domain of uniform convergence. With $g_{n\nu}(z)$ continuous and $g(z)$ uniformly convergent, both the differentiated and the integrated series are power series which again are continuous functions and have the same radius of convergence as the original series.

By use of Eqs. (3.31), (3.16), and (3.14) and performing some trivial mathematical manipulations we can express deflection, slope, bending moment and shearing force as

$$g(\ell;\, z) = \sum_{k=1}^{4} \gamma_k g(\ell,\, k;\, z), \quad \ell = 1, 2, 3, 4, \tag{3.33}$$

where the parameter ℓ refers to the following physical functions:

$\ell = 1 \ldots g(z)$; deflection

$\ell = 2 \ldots \frac{d}{dz} g(z)$; slope

$\ell = 3 \ldots \left[\left((z + 1)^4 - 1\right) \frac{d^2}{dz^2} g(z)\right]$; modified bending moment

$$\ell = 4 \ldots \frac{d}{dz} \left[\left((z + 1)^4 - 1\right) \frac{d^2}{dz^2} g(z)\right] \text{; modified shearing force.} \tag{3.34}$$

The γ_k are the coefficients of the four linearly independent solutions $g(1,\, k;\, z)$, $k = 1, 2, 3, 4$, and must be determined to within an arbitrary constant in each particular case from the boundary conditions. The four linearly independent solutions and their modified derivatives are given by

$$g(\ell, k; z) = \sum_{n=0}^{\infty} c(\ell, k, n)z^n + \delta_{4k}g(\ell, 2; z)\ln z, \tag{3.35}$$

where the coefficients $c(\ell, k, n)$ are defined as

$$c(\ell, k, -n) = 0 \text{ for all } n = 0, 1, 2, 3, \ldots \tag{3.36}$$

$$c(1, 1, 0) = 1; \quad c(1, 2, 0) = c(1, 3, 0) = c(1, 4, 0) = 0$$

$$c(1, 1, 1) = 0; \quad c(1, 2, 1) = 1; \quad c(1, 3, 1) = c(1, 4, 1) = 0$$

$$c(1, 1, 2) = c(1, 2, 2) = 0; \quad c(1, 3, 2) = 1; \quad c(1, 4, 2) = 0$$

$$\begin{aligned} c(1, k, n) = -\frac{1}{n(n-1)} \Bigg\{ \Bigg[&\frac{3}{2}(n-1)(n-2)c(1, k, n-1) + (n-2) \\ &\times (n-3)c(1, k, n-2) + \frac{1}{4}(n-3)(n-4)c(1, k, n-3) \\ &- \frac{1}{(n-1)(n-2)}\left(\frac{\omega^4}{4}\right)\Big(2c(1, k, n-4) + c(1, k, n-5)\Big)\Bigg] \\ &+ \delta_{4k}\Bigg[(2n-1)\, c(1, 2, n-1) + \frac{3}{2}(2n-3) \\ &\times c(1, 2, n-2) + (2n-5)\, c(1, 2, n-3) + \frac{1}{4}(2n-7) \\ &\times c(1, 2, n-4) + \frac{2n-3}{(n-1)^2(n-2)^2}\left(\frac{\omega^4}{4}\right)\Big(2c(1, k, n-5) \\ &+ c(1, k, n-6)\Big)\Bigg]\Bigg\}; \quad n = 3, 4, 5, \ldots \end{aligned} \tag{3.36.1}$$

$$c(2, k, n) = nc(1, k, n + 1) + \delta_{4k}c(1, 2, n + 1); \text{ for all } n=0,1,2,3,\ldots \tag{3.36.2}$$

$$c(3, 1, 0) = c(3, 2, 0) = c(3, 3, 0) = 0; \quad c(3, 4, 0) = 4$$

$$c(3, 1, 1) = c(3, 2, 2) = 0; \quad c(3, 3, 2) = 8; \quad c(3, 4, 2) = 6$$

$$c(3, k, n) = \frac{\omega^4}{n(n - 1)} \Bigg\{ [2c(1, k, n - 3) + c(1, k, n - 4)]$$

$$-\delta_{4k} \frac{2n - 1}{n(n - 1)} [2c(1, 2, n - 3) + c(1, 2, n - 4)] \Bigg\};$$

$$n = 2, 3, 4, \ldots \tag{3.36.3}$$

$$c(4, 1, 0) = c(4, 2, 0) = 0; \quad c(4, 3, 0) = 8; \quad c(4, 4, 0) = 6$$

$$c(4, k, n) = \left(\frac{\omega^4}{n}\right) \Bigg\{ [2c(1, k, n - 2) + c(1, k, n - 3)]$$

$$-\delta_{4k} \frac{1}{n} [2c(1, 2, n - 2) + c(1, 2, n - 3)] \Bigg\};$$

$$n = 1, 2, 3, \ldots \tag{3.36.4}$$

and δ_{4k} is the Kronecker delta function $\left(\delta_{mn} = 0 \text{ for } m \neq n \text{ and } \delta_{mn} = 1 \text{ for } m = n\right)$.

4. FREE LATERAL VIBRATIONS OF A CONICAL BALLISTIC TUBE WITH A CYLINDRIC BORE

The determination of the eigenvalues of the separation constant $a = k\omega^2$ and the relative values of the coefficients γ_1, γ_2, γ_3, and γ_4 are contingent on the boundary conditions at the ends of the beam. Let us assume that the solution of Eq. (3.2), $g(1, z) = \sum_{k=1}^{4} \gamma_k \, g\left(1, k; \omega^4; z\right)$,

is constrained

$$\text{at } z = z_r \text{ by } g(\ell_1, z_r) = g(\ell_2, z_r) = 0, \ \ell_1 \neq \ell_2, \text{ and} \tag{4.1.1}$$

$$\text{at } z = z_\ell \text{ by } g(\ell_3, z_\ell) = g(\ell_4, z_\ell) = 0, \ \ell_3 \neq \ell_4, \tag{4.1.2}$$

where ℓ_1, ℓ_2, ℓ_3, and ℓ_4 are any of the physical functions given in Eq. (3.34). In matrix formulation, the above boundary value equations can be written as

$$\begin{vmatrix} g\left(\ell_1,1;\ z_r;\ \omega^4\right) & g\left(\ell_1,2;\ z_r;\ \omega^4\right) & g\left(\ell_1,3;\ z_r;\ \omega^4\right) & g\left(\ell_1,4;\ z_r;\ \omega^4\right) \\ g\left(\ell_2,1;\ z_r;\ \omega^4\right) & g\left(\ell_2,2;\ z_r;\ \omega^4\right) & g\left(\ell_2,3;\ z_r;\ \omega^4\right) & g\left(\ell_2,4;\ z_r;\ \omega^4\right) \\ g\left(\ell_3,1;\ z_\ell;\ \omega^4\right) & g\left(\ell_3,2;\ z_\ell;\ \omega^4\right) & g\left(\ell_3,3;\ z_\ell;\ \omega^4\right) & g\left(\ell_3,4;\ z_\ell;\ \omega^4\right) \\ g\left(\ell_4,1;\ z_\ell;\ \omega^4\right) & g\left(\ell_4,2;\ z_\ell;\ \omega^4\right) & g\left(\ell_4,3;\ z_\ell;\ \omega^4\right) & g\left(\ell_4,4;\ z_\ell;\ \omega^4\right) \end{vmatrix} \begin{vmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{vmatrix} \tag{4.2}$$

A non-trivial solution of Eq. (4.2) exists only if the determinant of the coefficients is equal to zero

$$\begin{vmatrix} g\left(\ell_1,1;\ z_r;\ \omega^4\right) & g\left(\ell_1,2;\ z_r;\ \omega^4\right) & g\left(\ell_1,3;\ z_r;\ \omega^4\right) & g\left(\ell_1,4;\ z_r;\ \omega^4\right) \\ g\left(\ell_2,1;\ z_r;\ \omega^4\right) & g\left(\ell_2,2;\ z_r;\ \omega^4\right) & g\left(\ell_2,3;\ z_r;\ \omega^4\right) & g\left(\ell_2,4;\ z_r;\ \omega^4\right) \\ g\left(\ell_3,1;\ z_\ell;\ \omega^4\right) & g\left(\ell_3,2;\ z_\ell;\ \omega^4\right) & g\left(\ell_3,3;\ z_\ell;\ \omega^4\right) & g\left(\ell_3,4;\ z_\ell;\ \omega^4\right) \\ g\left(\ell_4,1;\ z_\ell;\ \omega^4\right) & g\left(\ell_4,2;\ z_\ell;\ \omega^4\right) & g\left(\ell_4,3;\ z_\ell;\ \omega^4\right) & g\left(\ell_4,4;\ z_\ell;\ \omega^4\right) \end{vmatrix} = 0 \tag{4.3}$$

This equation furnishes the means for the determination of the eigenvalues $\left(\omega^4\right)_n$ and, consequently, from Eq. (4.2) the ratios of the coefficients γ_2/γ_1, γ_3/γ_1 and γ_4/γ_1 and from Eq. (3.33) the eigenfunctions $g_n(1;z)$ as well.

The roots of the characteristic Eq. (4.3) represent the eigenvalues of the system and correspond to the natural frequencies of the beam. Using the boundary conditions, Eq. (4.1), and the orthogonality relation for the eigenfunctions, one can show that the eigenvalues $\left(\omega^4\right)_n$ must be positive or zero.[2] Due to the complexity of the functions $g(\ell,\ k;\ z;\ \omega^4)$ the eigenvalues $\left(\omega^4\right)_n$ cannot be determined analytically but must be found numerically.

As a numerical example, we will consider the eigenvalue problem of the free transverse vibration of a circular conical tube with a circular cylindric bore where the bottom of the cone is a fixed end and the top is allowed to vibrate freely (Figure 4.1) and compare the solution to this problem with the free transverse vibrations of a circular tube.

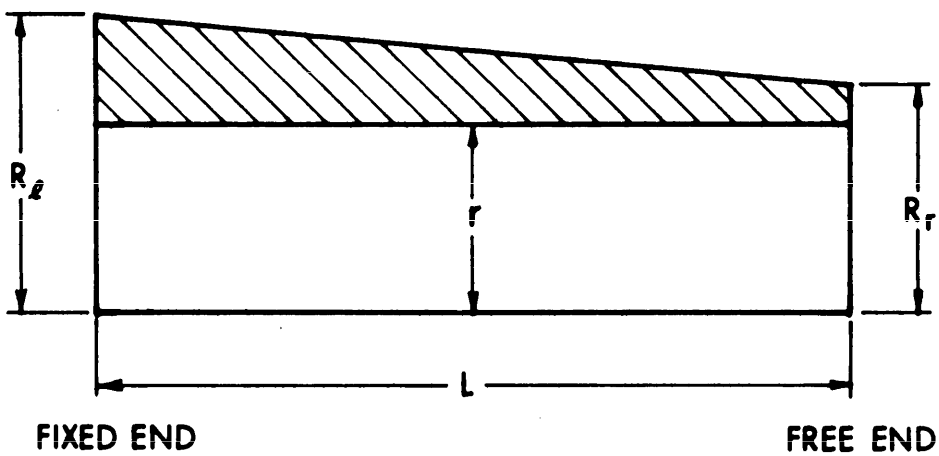

Figure 4.1. Geometry of Numerical Example

We select steel with a density of 7.84×10^3 kg/m^3 and a Young's modulus of elasticity of 2.1×10^{11} Pa as the tube material, a beam length of 1.00m, an interior radius of 0.05m and for the circular pipe an exterior radius of 0.06m and determine the exterior radii R_r and R_ℓ by keeping the volume constant

[2] *A. S. Elder, "Free and Forced Vibrations of a Tapered Cantilever Beam," University of Delaware; M. A. Thesis, June 1956.*

$$V_{h.\ cone} = \pi \ell \left[\frac{1}{3}\left(R_\ell^2 + R_\ell R_r + R_r^2\right) - r^2\right] = V_{pipe} = \pi \ell \left(R^2 - r^2\right) \tag{4.4}$$

and varying R_r.

$$R_\ell = \sqrt{3\left[R^2 - \left(\frac{R_r}{2}\right)^2\right]} - \left(\frac{R_r}{2}\right); \quad R_r \text{ and } R_\ell \geq r. \tag{4.5}$$

Tables 4.1, 4.2, and 4.3 are representative samples of the obtained numerical results.

TABLE 4.1. NATURAL FREQUENCIES OF A CONCENTRIC CIRCULAR CONE WITH A CYLINDRIC BORE AS FUNCTION OF THE MODE OF VIBRATION FOR R = 0.95

n	Ω_n	ω_n	Δ_n
1	9.23175	6.32619	.45929
2	18.66643	15.83683	.17867
3	28.86741	26.50023	.08933
4	39.18171	37.09654	.05621
5	49.56180	47.69570	.03913
6	59.98144	58.29474	.02893
7	70.42839	68.89379	.02227
8		79.49283	

TABLE 4.2. NATURAL FREQUENCIES OF A CONCENTRIC CIRCULAR CONE WITH A CYLINDRIC BORE AS FUNCTION OF THE MODE OF VIBRATION FOR R = -0.95

n	Ω_n	ω_n	Δ_n
1	3.67169	6.32619	-.41960
2	13.36686	15.83683	-.15596
3	24.81891	26.50023	-.06345
4	35.66632	37.09654	-.03855
5	46.41514	47.69570	-.02685
6	57.11542	58.29474	-.02023
7		68.89379	
8		79.49283	

TABLE 4.3. NATURAL FREQUENCIES OF A CONCENTRIC CIRCULAR CONE WITH A CYLINDRIC BORE AS FUNCTION OF VARIOUS RADII

$\|R\|$	$\Omega_1(+\|R\|)$	ω_1	$\Omega_1(-\|R\|)$
1.		6.01764	
.90233		6.66899	
.80417	10.15110	7.48786	5.16764
.70550		8.52967	
.60631	12.37331	9.92501	7.75756
.50661		11.87835	
.40637	17.10772	14.80818	12.69508
.30560		19.69097	
.20429	31.65520	29.45615	27.34957
.10243		58.75086	

Tables 4.1 and 4.2 contain the first eight natural frequencies, Ω_n, of a concentric circular cone with a concentric circular cylindric bore for two selected values of R_ℓ and R_r expressed by the ratio $R = \left(R_\ell - R_r\right)/\max\left(R_\ell - R_r\right)$. The ω_n's are the corresponding eigenvalues for the circular tube. The quantity $\Delta_n = |\Omega_n - \omega_n|/\omega_n$ expresses the deviation of Ω_n from ω_n. Table 4.3 exhibits the dependence of the first natural frequency on the radii of the cone.

The implication of the numerical result to the geometrical description of gun tubes for lateral tube motion can best be seen from Figure 4.2. Here we plotted the deviation of the natural frequencies of the concentric circular cone with a concentric circular cylindric bore from the circular tube versus the normalized difference between the left and right radii of the cone. The plot displays clearly that for eigenfrequencies in the lower principal mode of vibration the commonly used approximation of a conical gun tube by a cylindric beam is not justified. However, for the higher principal modes of lateral vibration we can well substitute the eigenvalues of the beam for the correct ones.

Having the natural frequencies determined we can use Eq. (3.33) to compute the deflections of the centerline of the concentric circular cone with a concentric cylindric bore in the r-th mode. Figure 4.3 is a representative example. The values on the abscissa correspond to the deflections of the beam axis normalized by the maximum displacement and on the ordinate to the axial distance of the beam from the left boundary normalized by the layer of the beam.

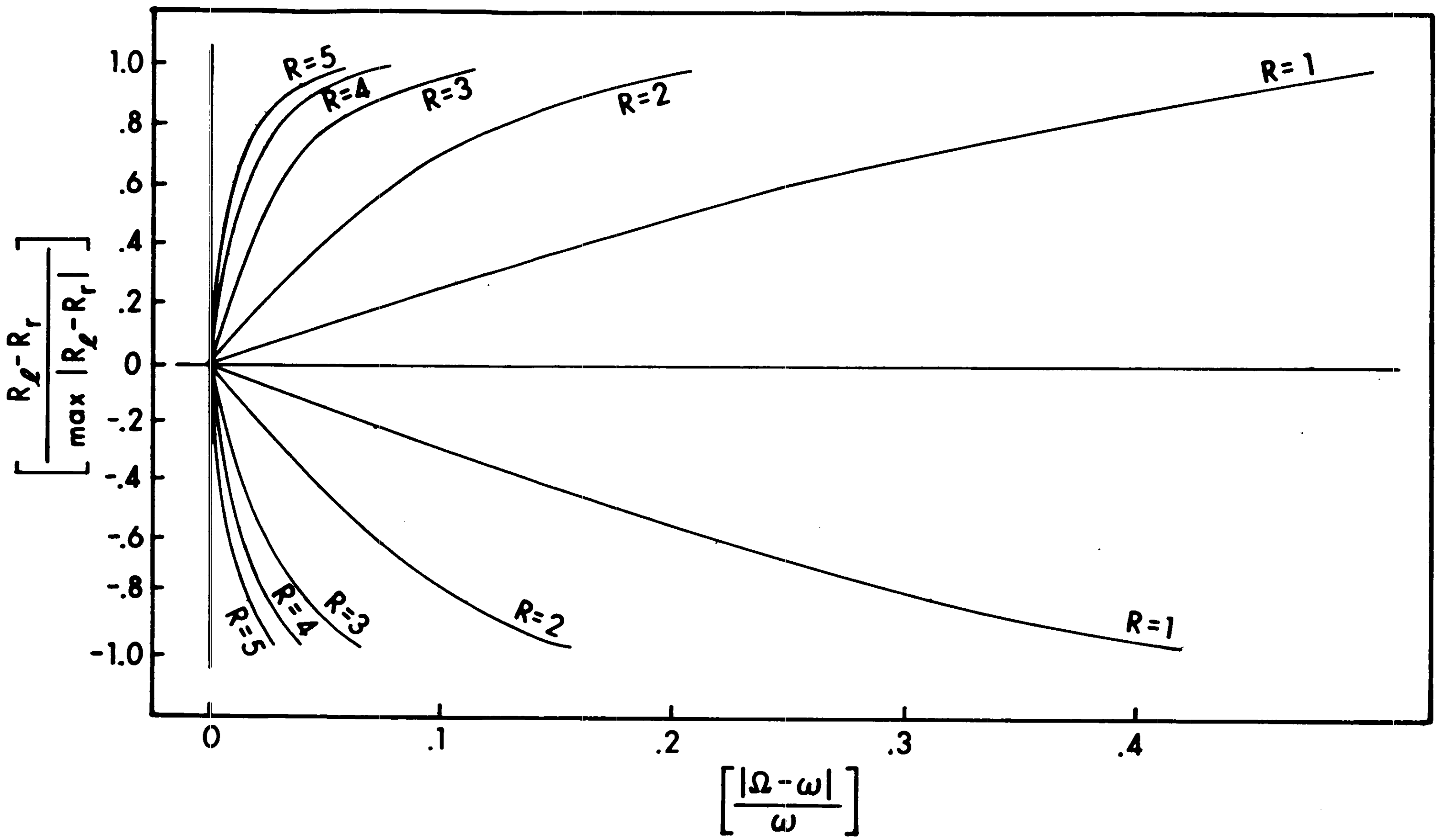

Figure 4.2. Deviation of the Natural Frequencies of a Concentric Circular Conic Beam with a Concentric Circular Cylindric Bore From those of an Equivalent Circular Tube as Function of the Cone Radii

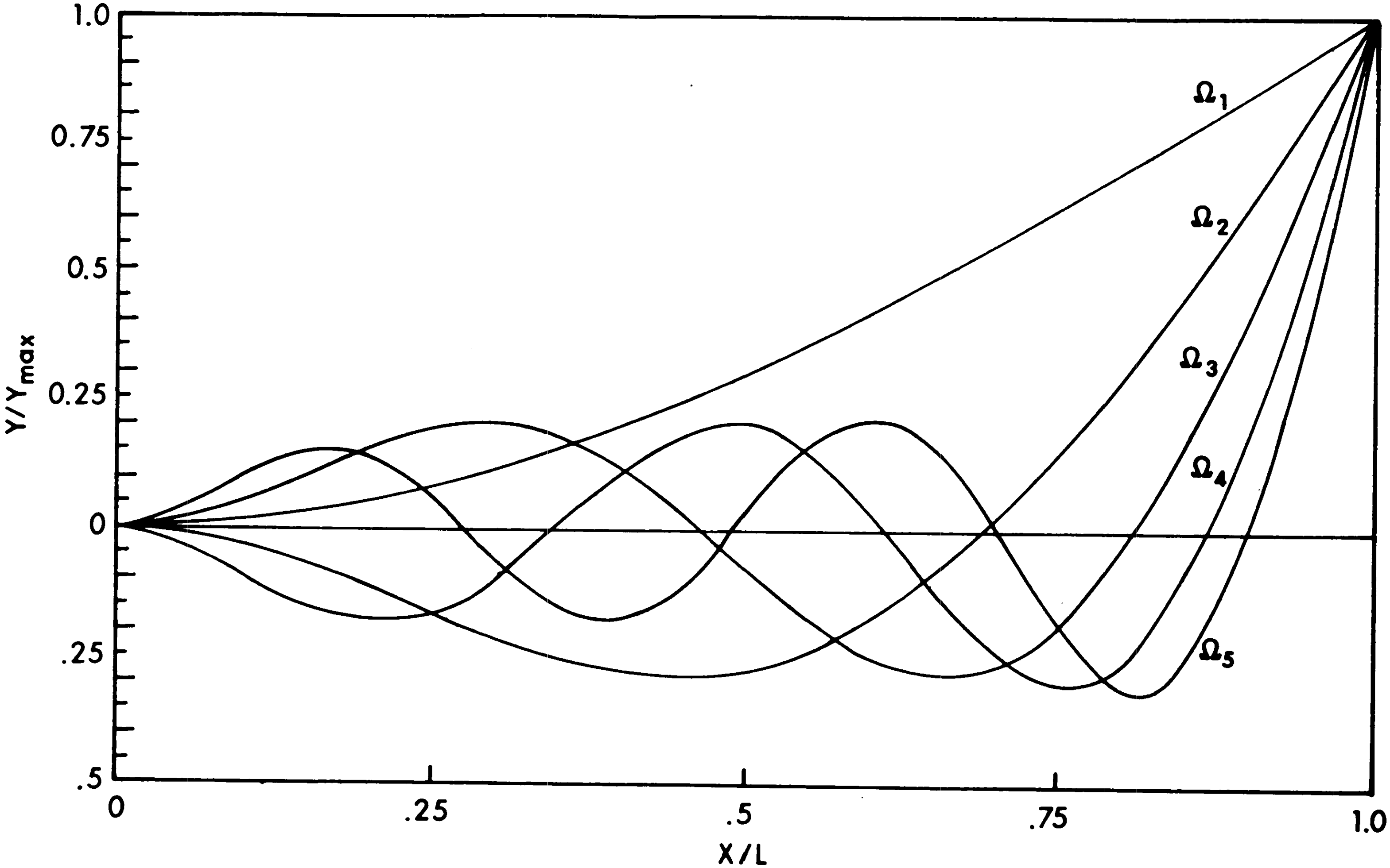

Figure 4.3. Deflection of a Concentric Circular Conic Beam with a Concentric Circular Cylindric Bore Versus Axial Distance for the First Five Principal Modes of Lateral Vibration for R = .95123

ACKNOWLEDGEMENTS

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